4.2 Null Space - Column Space and Linear Transformation

Consider the two matrices

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 3 \\ 1 & 3 & -3 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix}$$

We are interested in two sets:

1) Set of Solutions of $B\vec{x} = \vec{0}$.

2) Set of all linear combinations of the colums of A.

In our case, we can obtain the solutions of

$$B\vec{x} = \vec{0}$$

Considering the augmented matrix and row-reducing it

$$\left[\begin{array}{ccccc} 1 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc} 1 & 0 & -2 & 4 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{array}\right]$$

$$\Rightarrow \left.\begin{array}{l} x_1 = 2x_3 - 4x_4 \\ x_2 = -3x_3 + 2x_4 \\ x_3, x_4 \text{ free Variables} \end{array}\right\} \Rightarrow$$

Set of Solns

To find the set of Solutions

1) We express the Solns. as

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} 2x_3 - 4x_4 \\ -3x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

x_3 and x_4 free variables.

2) Since x_3 and x_4 are arbitrary, the set of Solns. of $B\vec{x} = \vec{0}$ is given by

$$N = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : \vec{x} = c_1 \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \ c_1, c_2 \in \mathbb{R} \right\}$$

It means all possible linear combinations of

$$\vec{v}_1 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

In other words,

$$N = \text{Span} \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

And N is a Vector space or a Subspace of $\mathbb{R}^4$.

This particular result is true in general.

<u>Definition</u> The set of solutions of

$$A\vec{x}=\vec{0}$$

where $A_{m\times n}$ matrix is called <u>null space</u> of A. In set notation

$$\text{Nul } A = \{ \vec{x}\in\mathbb{R}^n : A\vec{x}=\vec{0}\}.$$

<u>Thm 2.-</u> The null space of $A_{m\times n}$ is a subspace of $\mathbb{R}^n$.

<u>Proof.-</u> First notice that any soln. $\vec{x}$ of $A\vec{x}=\vec{0}$ is in $\mathbb{R}^n$, because A has n columns.

Therefore $\text{Nul } A \subset \mathbb{R}^n$.

$\text{Nul } A$ is nonempty because $\vec{0}\in \text{Nul } A$, since

$$A\vec{0}=\vec{0}$$

On the other hand,

If $\vec{u},\vec{v}\in \text{Nul}A \Rightarrow A(\vec{u}+\vec{v}) \overset{\text{Matrix multiplication}}{=} A\vec{u}+A\vec{v} = \vec{0}+\vec{0}=\vec{0}$

If $c\in\mathbb{R}, \vec{u}\in \text{Nul}A \Rightarrow A(c\vec{u}) \overset{\text{Matrix Multipl.}}{=} cA\vec{u} = c\vec{0}=\vec{0}.$

Therefore, $\left.\begin{array}{l} \text{If } \vec{u},\vec{v}\in \text{Nul } A \Rightarrow (\vec{u}+\vec{v})\in \text{Nul } A \\ \text{If } c\in\mathbb{R}, \vec{u}\in\text{Nul}A \Rightarrow (c\vec{u})\in \text{Nul } A \end{array}\right\} \Rightarrow \text{Nul } A$ is a subspace.

from A.

Now, we will consider the vector space

$$\text{Col} A = \text{Span}\{\text{Columns of } A\}$$

In our example,

$$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -3 \\ 1 \end{bmatrix} \right\}$$

What does col A has to do with the System

$$A\vec{x} = \vec{b} \; ?$$

<u>Def.</u>- The Column space of a matrix $A_{m\times n}$ (Col A) is the set of all linear Combinations of the Columns of A.

For $A = [\vec{a}_1, \vec{a}_2 \ldots \vec{a}_n]$

$$\text{Col } A = \text{Span}\{\vec{a}_1, , \vec{a}_n\}$$

By definition, the following theorem is trivially true

<u>Thm 3.</u>- The Column Space of $A_{m\times n}$ matrix is a Subspace of $\mathbb{R}^m$.

Exercise #16

Find $A_{m \times n}$ such that the given set is Col A.

$$H = \left\{ \vec{y} = \begin{bmatrix} b-c \\ 2b+3d \\ b+3c-3d \\ c+d \end{bmatrix} : b, c, d \text{ real} \right\}$$

The procedure is to Separate $\vec{y}$ in three Vectors and multiply them by the scalars a, b, c as follows:

$$\begin{bmatrix} b-c \\ 2b+3d \\ b+3c-3d \\ c+d \end{bmatrix} = \begin{bmatrix} b \\ 2b \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} -c \\ 0 \\ 3c \\ c \end{bmatrix} + \begin{bmatrix} 0 \\ 3d \\ -3d \\ 1 \end{bmatrix}$$

$$= b \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix} + d \begin{bmatrix} 0 \\ 3 \\ -3 \\ 1 \end{bmatrix}$$

Then

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 3 \\ 1 & 3 & -3 \\ 0 & 1 & 1 \end{bmatrix}$$ is such that

Col $A = H$.

Work the way back: Given A find H.

Linear Transformation: Kernel and range.

Def.—
$$T: V \longrightarrow W$$
$$\vec{x} \longrightarrow T(\vec{x})$$

is a linear transformation if and only if

i) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$, for all $\vec{u}, \vec{v} \in V$

ii) $T(c\vec{u}) = cT(\vec{u})$, for all $\vec{u} \in V$ and $c \in \mathbb{R}$.

Def.—

i) Kernel of T: Set $K \subset V$ Such that $\vec{x} \in K$ if and only if $T(\vec{x}) = \vec{0}$.

ii) range of T: Set $H \subset W$ Such that $\vec{b} \in H$ if and only if there exists $\vec{x} \in V$ and $T(\vec{x}) = \vec{b}$.

#32 $T: \mathbb{P}_2 \longrightarrow \mathbb{R}^2$

$$T(\vec{p}) = \begin{bmatrix} p(0) \\ p(0) \end{bmatrix} \quad \text{or} \quad T(\overbrace{a_0 + a_1 t + a_2 t^2}^{\vec{p}}) = \begin{bmatrix} a_0 \\ a_0 \end{bmatrix} \overset{\vec{p}(0)}{=} a_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Kernel of T = any $\vec{p} \in \mathbb{P}_2$ s.t. $T(\vec{p}) = \begin{bmatrix} a_0 \\ a_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$$\Rightarrow \text{Kernel } T = \{\vec{p} = p(t) = a_1 t + a_2 t^2,\ a_1, a_2 \in \mathbb{R}\}.$$
$$= \text{Span}\{p_1(t) = t,\ p_2(t) = t^2\}.$$

Section 4.2

#32

Define a Linear transformation

$$T: \mathbb{P}_2 \longrightarrow \mathbb{R}^2$$

$$\left\{\begin{array}{l}\vec{p}(t) = p(t) \\ = a_0 + a_1 t + a_2 t^2\end{array}\right\} \longrightarrow \begin{bmatrix} \vec{p}(0) \\ \vec{p}(0) \end{bmatrix} = \begin{bmatrix} a_0 \\ a_0 \end{bmatrix}$$

Verify this transformation is linear

For example, if $\vec{p}$ is such that $\vec{p}(t) = p(t) = \underset{a_0}{\underset{\parallel}{3}} \underset{a_1=-1}{-t} + \overset{a_2=2}{2}t^2$

then, $$T(\vec{p}) = \begin{bmatrix} a_0 \\ a_0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Find polynomials $\vec{p}_1$ and $\vec{p}_2$ in $\mathbb{P}_2$ that Span the Kernel of T.

By definition, Kernel of $T = \left\{ \vec{p} \in \mathbb{P}_2 : T(\vec{p}) = \vec{0} \right\}$

For a given $\vec{p} \in \mathbb{P}_2$ We know $\vec{p}(t) = p(t) = a_0 + a_1 t + a_2 t^2$

and $T(\vec{p}) = \begin{bmatrix} a_0 \\ a_0 \end{bmatrix}$, So, for $T(\vec{p}) = \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

the coefficient $a_0 = 0$.

and Kernel of $T = \left\{ \vec{p} \in \mathbb{P}_2 : \vec{p}(t) = p(t) = a_1 t + a_2 t^2, \ a_1, a_2 \text{ arbitrary} \right\}$

So $\vec{p}$ is a linear combination of $\vec{p}_1(t) = p_1(t) = t$ and

$\vec{p}_2(t) = p_2(t) = t^2$.

It can be observed that

$$\text{Kernel of } T = \text{Span}\left\{ \vec{p}_1(t) = p(t) = t,\ \vec{p}_2(t) = p_2(t) = t^2 \right\}$$

Since a_1, a_2 are arbitrary.

Clearly,

$$\text{range of } T = \left\{ \vec{b} = \begin{bmatrix} a_0 \\ a_0 \end{bmatrix} : a_0 \text{ is arbitrary in } \mathbb{R} \right\}.$$

It can also be said that

$$\text{range of } T = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$